WBMA063-05 – Numerical Linear Algebra
Exam 2025-2026
Wednesday 28 February 09:00-11:00

Instructions:

1. **Write your name and student number of the top of each sheet of writing paper!**
2. Use the writing (lined) and scratch (blank) paper provided, raise your hand if you need more paper.
3. Start each question on a new page.

This exam consists of 4 questions for a total of 90 points. 10 points are free.

Question 1: 30 points

1. (a) (6 points) Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 10^{-5} & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Compute the LU decomposition (without pivoting) and the pivoted LU decomposition of the matrix A. You need not solve the linear system.

(b) (8 points) Consider the following algorithm for solving $A\boldsymbol{x} = \boldsymbol{b}$:

1. Compute the LU decomposition of A ($A = LU$) (no pivoting).
2. Solve $L\boldsymbol{y} = \boldsymbol{b}$ with forward substitution.
3. Solve $U\boldsymbol{x} = \boldsymbol{y}$ with backward substitution.

Suppose this algorithm is performed in floating point arithmetic and the LU decomposition is computed with backward error δA_{LU}:

$$\hat{L}\hat{U} = A + \delta A_{\text{LU}},$$

where $\hat{L}$ and $\hat{U}$ are the computed factors. Assume $\|\hat{L}\| = \mathcal{O}(\|L\|)$ and $\|\hat{U}\| = \mathcal{O}(\|U\|)$. Show that the computed solution $\hat{\boldsymbol{x}}$ satisfies

$$(A + \delta A)\hat{\boldsymbol{x}} = \boldsymbol{b}, \quad \|\delta A\| = \mathcal{O}(\|\delta A_{\text{LU}}\| + \varepsilon\|L\|\|U\|),$$

where $\varepsilon = \varepsilon_{\text{machine}}$ is machine precision. You may use the fact that triangular solves are backward stable.

(c) (5 points) In floating point arithmetic, we have the following statements for the backward error of the the computed factors $\hat{L}$ and $\hat{U}$:

$$\text{unpivoted LU:} \quad \hat{L}\hat{U} = A + \delta A_{\text{LU}}, \quad \|\delta A_{\text{LU}}\| = \mathcal{O}(\varepsilon\|L\|\|U\|)$$
$$\text{pivoted LU:} \quad \hat{L}\hat{U} = PA + \delta A_{\text{PLU}}, \quad \|\delta A_{\text{PLU}}\| = \mathcal{O}(\varepsilon\|L\|\|U\|),$$

where $\varepsilon = \varepsilon_{\text{machine}}$ is machine precision. This means that regardless of whether unpivoted or pivoted LU is used, we find that the backward error of the computed solution $\hat{\boldsymbol{x}}$ satisfies the following:

$$(A + \delta A)\hat{\boldsymbol{x}} = \boldsymbol{b}, \quad \|\delta A\| = \mathcal{O}(\varepsilon\|L\|\|U\|).$$

You need not show this. Explain why the pivoted LU decomposition is still preferable over the unpivoted LU decomposition when the linear system $A\boldsymbol{x} = \boldsymbol{b}$ in (a) is solved in floating point arithmetic. Specifically use the LU factors you found in (a) in your answer.

(d) (3 points) Another reason that the pivoted LU decomposition is preferable for a general linear system $A\boldsymbol{x} = \boldsymbol{b}$, is that the unpivoted LU decomposition may not exist. Give an example of a **non-singular** matrix A that does not allow an unpivoted LU decomposition. It is sufficient to simply provide A, you need not prove the LU decomposition does not exist.

(e) (8 points) Show that every non-singular matrix $A \in \mathbb{R}^{n\times n}$ has a pivoted LU decomposition.

Question 2: 24 points

2. Let $A \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix with real eigenvalues λ_i with distinct magnitudes:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

Consider the basic form of the QR algorithm:

$$\begin{aligned}
&A_1 = A \\
&\text{For } k = 1, 2, 3, \ldots \\
&\qquad Q_k R_k = A_k \quad \text{(QR decomposition)} \\
&\qquad A_{k+1} = R_k Q_k
\end{aligned}$$

(a) (5 points) We define

$$\begin{aligned}
Q^{(k)} &:= Q_1 Q_2 \ldots Q_k. \\
R^{(k)} &:= R_k R_{k-1} \ldots R_1.
\end{aligned}$$

Show that the QR decomposition of A^k (that is, A to the power k) is given by

$$A^k = Q^{(k)} R^{(k)}. \tag{1}$$

Hint: you may use the identity $A_{k+1} = \left[Q^{(k)}\right]^T A Q^{(k)}$ for $k \geq 1$ without proof.

(b) (6 points) Use (1) to show that, for some scalar α,

$$\left(A^{-k}\right)^T \boldsymbol{e}_n = \alpha \boldsymbol{q}_n^{(k)},$$

where $\boldsymbol{e}_n = [0, \ldots, 0, 1]^T \in \mathbb{R}^n$ is the nth canonical basis vector and $\boldsymbol{q}_n^{(k)}$ is the last column of $Q^{(k)}$.

(c) (6 points) Let λ_n be the eigenvalue of A with smallest magnitude as above. Show that

$$\boldsymbol{q}_n^{(k)} \to \pm \boldsymbol{w}_n \quad \text{as } k \to \infty, \quad \text{where} \quad A^T \boldsymbol{w}_n = \lambda_n \boldsymbol{w}_n, \tag{2}$$

for a particular vector $\boldsymbol{w}_n$ with unit norm, $\|\boldsymbol{w}_n\|_2 = 1$.

(d) (7 points) From the identity in (c), we can show that the last row of the iterates A_k converges to

$$\begin{bmatrix} 0 & \ldots & 0 & \lambda_n \end{bmatrix}$$

as $k \to \infty$. (You need not prove this). This is the basis of the QR algorithm. The actual QR algorithm that is implemented has more bells and whistles. The eigenvalues are found one-by-one, and each time an eigenvalue is found the dimension of the problem is reduced. Explain this algorithm. In particular, mention how the convergence to a single eigenvalue can be sped up, and mention how the dimension is reduced and a new eigenvalue can be found.

Question 3: 14 points

3. Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$ where $A = A^T \in \mathbb{R}^{n\times n}$ is symmetric positive definite.

   (a) (4 points) One may approximate a solution to the symmetric positive definite system $A\boldsymbol{x} = \boldsymbol{b}$ using the Conjugate Gradients (CG) algorithm. The kth CG iteration $\boldsymbol{x}_k \in \mathbb{R}^n$ is the solution to a minimization problem. What minimization problem is that? Define all notation you use. Assume $\boldsymbol{x}_0 = \boldsymbol{0}$.

   (b) (10 points) Let $A = V\Lambda V^T$ be the eigendecomposition of A. Show that the kth CG iterate $\boldsymbol{x}_k$ satisfies

$$\|\boldsymbol{x}^* - \boldsymbol{x}_k\|_A = \min_{p\in\mathcal{P}_k} \|Vp(\Lambda)V^T\boldsymbol{x}^*\|_A,$$

   where $\mathcal{P}_k$ is the set of all degree k polynomials p with $p(0) = 1$ and $\boldsymbol{x}^*$ is the solution to the linear system. Again assume $\boldsymbol{x}_0 = \boldsymbol{0}$.

Question 4: 22 points

4. Consider a matrix $A \in \mathbb{R}^{m\times n}$ with $m > n$. Assume A has full column rank. A QR decomposition of A is given by $A = QR$, where $Q \in \mathbb{R}^{m\times n}$ has orthonormal columns and $R \in \mathbb{R}^{n\times n}$ is upper triangular.

   (a) (10 points) A Householder reflector $H \in \mathbb{R}^{m\times m}$ is a symmetric and orthogonal matrix given by

$$H = I - 2\boldsymbol{v}\boldsymbol{v}^T,$$

   where $\boldsymbol{v} \in \mathbb{R}^m$ is a vector with unit norm, i.e. $\|\boldsymbol{v}\|_2 = 1$. A special property is that for two vectors $\boldsymbol{u}, \boldsymbol{w} \in \mathbb{R}^m$ with the same norm ($\|\boldsymbol{u}\|_2 = \|\boldsymbol{w}\|_2$), we can choose

$$\boldsymbol{v} = \frac{\boldsymbol{u} - \boldsymbol{w}}{\|\boldsymbol{u} - \boldsymbol{w}\|_2}$$

   in $H = I - 2\boldsymbol{v}\boldsymbol{v}^T$, to obtain

$$H\boldsymbol{u} = \boldsymbol{w} \quad \text{and} \quad H\boldsymbol{w} = \boldsymbol{u}.$$

   One approach to computing a QR decomposition of A is to use Householder reflectors. Explain this algorithm. Be specific about the various Householder reflectors that you apply. Specify what the final matrices Q and R are.

   (b) (12 points) An alternative algorithm to computing a QR decomposition could be to use Gram-Schmidt orthogonalization. Explain how you could compute a QR decomposition using Gram-Schmidt and provide the flop-count of your algorithm.

END OF EXAM